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Some Theory and Results for Metrics for Bounded Response Scales

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In an earlier note, a new metric for bounded response scales (MBR) was introduced which resembles the city-block metric but is bounded above. It was suggested the MBR may be more appropriate than minkowski metrics for data obtained with bounded response scales. In this article, some formal properties of the MBR are investigated and it is shown that it is indeed a metric. Empirical predictions are then derived from the MBR and contrasted with those of a "monotonicity hypothesis," which holds that dissimilarity judgements tend to be biased towards overestimation of larger distances, and with the predictions of the minkowski metrics, which imply additivity of collinear segments. Some empirical results are presented which contradict the monotonicity hypothesis and the minkowski metrics, and favor the MBR. Finally, the logic used to motivate the MBR is invoked to define a subadditive concatenation for bounded norms in the one-dimensional case, which may be useful in psychophysical work where the upper bounds are often real, rather than due to the response scale. This concatenation predicts underestimation for doubling and overestimation for halving and middling tasks.

1. INTRODUCTION

Most modern theories of psychological measurement are predicated on the Archimedean axiom which characterizes the natural numbers N: for all nonzero $p, q \in N$, with $p < q$, there exists a number $n \in N$ such that $np > q$. As Helmholtz (1887) already observed, this axiom is at the root of the connection between Zählen und Messen (counting and measuring, the title of his paper). By "Messen" he meant the extensive measurement of physics. Thus if a, b are two physical quantities for which a concatenation operation $*$ is defined, then the Archimedean axiom says that no matter how small a may be, and no matter how large b is, n-fold concatenation of a with itself will eventually exceed b. If a is sufficiently small, it therefore can be used to measure any $b > a$ by expressing b numerically relative to a.

An obvious consequence of the Archimedean axiom is that for unrestricted concatenation it rules out maximal elements. This condition seemed to be met in classical, Newtonian physics. However, when Michelson and Morely showed towards the end of the last century that it is not met for velocity, the previously used additive

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concatenation for velocity had to be replaced with a non-additive concatenation, which is formally equivalent to the addition rule for hyperbolic tangents.

Notwithstanding its central role in axiomatic measurement theories, "probably the most vexing of the axioms is the Archimedean one which, as was pointed out, may be trivially true or may be difficult to test" (Krantz, Luce, Syppes, & Tversky 1971, p. 130). Similarly, Luce and Marley (1969) feel that a "limitation of the traditional theory is its failure, even when there is no restriction on concatenation, to take into account the possibility that the system may have a maximal element. The best known example is velocity: according to the theory of relativity, no velocity may exceed the speed of light" (p. 236). Krantz *et al.* (1971) treat this well-known example as an instance of "non-additive representations" in a section entitled "Essential Maxima in Extensive Structures," which follows closely Luce and Marley (1969). Both treatments are purely formal and restrict the interpretation to physics. One point of the present paper is to draw attention to the fact that non-additive structures conflicting with the Archimedean axiom may also arise in psychology.

The Archimedean axiom has also found its way into multidimensional scaling (MDS). For example, all minkowski metrics are predicated on it. The preoccupation with this particular family of metrics began perhaps with Attneave (1950), who, in this groundbreaking paper, introduced many of the assumptions and research themes which guided MDS to its present stage of popularity. Among other things, Attneave wanted to test the "Householder–Landahl" hypothesis that the simple city-block metric, which, in the planar case, can be written

$$d_c = a + b, \tag{1.1}$$

described dissimilarity judgments of physical shapes, e.g., parallelograms which varied in width and tilt, better than the more familiar Euclidean metric

$$d_e = (a^2 + b^2)^{1/2}, \tag{1.2}$$

where in both cases a, b denote the lengths of the orthogonal projections of the line segment between two stimuli A, B onto two orthogonal coordinate axes, as in Fig. 1a. Since d_c makes less unreasonable requirements on the judgmental capabilities of the subjects than d_e, the gist of the Householder–Landahl hypothesis is that d_c is the more plausible metric for dissimilarity judgments when the stimuli are "analyzable," i.e., when the underlying subjective dimensions are evident upon inspection of the stimuli. In the above example, these underlying dimensions are presumably subjective width and subjective tilt.

Since Attneave found that the observed data were not additive along single dimensions, he transformed them by subtracting a positive constant estimated on the basis of one-dimensional comparisons. Thus he was perhaps the first investigator to invoke a monotone transformation to convert observed dissimilarity judgments into "distances":

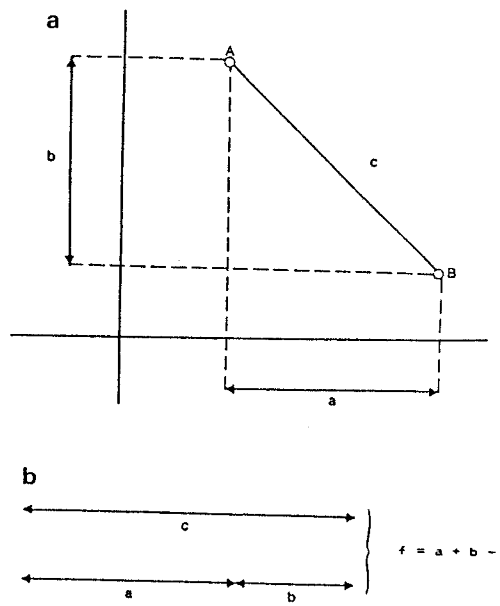

FIG. 1. Notation used for defining metrics and discriminant function $f = a + b - c$. (a) Notation used for defining minkowski metrics and MBR, (b) notation used for defining function f to discriminate between monotonicity hypothesis and MBR hypothesis.

> If these quantitities are to be considered "distances" in any meaningful sense, they should be additive along any single continuum ··· large but *strikingly consistent* discrepancies suggest that, if we moved our origin up by about 3.4 units, we might obtain quantities which would display, at least over a considerable range, the properties of distance. (Attneave, 1950, p. 524, my emphasis).

It might be noted in passing, that, strictly speaking, his observed data already were distances before the transformation, otherwise he could not have obtained distances by subtracting a positive constant (I am grateful to one of the reviewers for drawing my attention to this point). Thus the real motivation for invoking the monotone transformation was not, as he wrote, to convert his data into distances, but rather to convert them into distances which are segmentally additive. What is more important, however, is that Attneave gave no psychological reasons why his subjects consistently underestimated the "true" distances. From reading the article, it appears he simply transformed his data because they violated additivity of collinear line segments.

Subsequently, such "additive constant" transformations were often invoked to correct violations of the triangle inequality. Torgerson (1951; 1958, p. 208) analyzed "comparative distances" $h_{ij} = d_{ij} + c$ which he had derived from proportions obtained with the method of triads. He found he had to add $c = 3.60$ before his data approximately satisfied the triangle inequality.

For directly observed dissimilarity judgements, symmetry and nonnegativity are usually tautologically satisfied per instruction. Hence the only possible counterindication to the hypothesis that the observed dissimilarity judgments are distances are consistent violations of the triangle inequality. If the violations were essentially random, they could be ignored as error. Hence, to use an additive constant transformation to correct for consistent violations of the triangle inequality implies, in the absence of any other justification, a psychological monotonicity hypothesis which, as far as I know, has never been subjected to systematic empirical tests:

Monotonicity hypothesis. Dissimilarity judgments tend to be biased towards overestimation of larger distances, so that a positive monotone transformation is needed to restore the triangle inequality.

A second point of this paper, then, is to adduce some empirical evidence that this monotonicity hypothesis is often false, at least if bounded response scales are used, and that in those cases the monotonicity hypotheses can be replaced with an alternative hypothesis which invokes a *known* monotone transformation on psychological grounds. This alternative hypothesis predicts, to the contrary, that the triangle inequality is met even for collinear comparisons because larger distances are consistently underestimated, which is what Attneave found.

Upon further analysis, Attneave concluded that the Householder–Landahl hypotheses was borne out by his data: "Where metric treatment was possible, the composite or crossdimensional 'distances' between stimuli were found to be much greater than would be expected in a Euclidean space, but were approximately equal to the sum of the 'distances' along the basic dimensions psychological-reference systems, as the second hypothesis above would demand. These results unequivocally imply the existence of unique psychological reference-systems underlying the perceptions of similarity and differences among stimuli" (p. 555). As is well known, these early beginnings later grew into "non-metric" MDS, where the desired monotone transformation for carrying dissimilarities into distances is obtained iteratively on a computer (e.g., Kruskal, 1964), and where the Householder–Landahl hypothesis has been broadened to include all minkowski metrics,

$$d_r = (a^r + b^r)^{1/r}, \tag{1.3}$$

for all exponents $r \geqslant 1$. For $r = 2$, one obtains the Euclidean metric, for $r = 1$, the city-block metric, and as r approaches infinity, one obtains the so-called sup-metric. In the 1-dimensional case, all minkowski metrics coincide with the city-block metric which predicts segmental additivity along all lines parallel to the coordinate axes.

On the basis of a loss function ("stress"), the user was to decide which particular minkowski metric was appropriate for this data. This once widely popular research strategy is vulnerable to various criticisms. Criticisms of more philosophical nature can be found in Beals, Krantz, and Tversky (1968, p. 141). They argue that every metric in the last analysis implies a psychological model of some sort so that computational convenience alone should not be the ultimate criterion for selecting a metric. Technical criticisms were presented by Wolfrum (1976). She found, by theoretical analysis alone, that the "stress"-function is virtually flat for a large class of planar configurations, which makes it an unreliable tool for determining the minkowski exponent. Here we mention one other, seemingly obvious point which appears to have been widely overlooked: once one embarks on the quest for the psychologically correct metric, then there is no obvious reason to limit it to the minkowski metrics. This family of metrics, though "large," constitutes only a vanishing fraction of all conceivable metrics, i.e., of real value functions on pairs which satisfy the three distance axioms. Moreover, a priori there is nothing psychologically obvious about most minkowski metrics, with the possible exception of the city-block metric considered by Attneave.

A third point of this paper is, therefore, that most minkowski metrics make little sense psychologically, and that they can often be replaced with an alternative metric which does make some psychological sense and which approximates the minkowski metrics very closely in a sense yet to be specified.

2. Metrics for Bounded Response Scales

Even the city-block metric becomes suspect as a model for dissimilarity judgments when the rating scales used to obtain the numerical assignments are bounded above. For example, if we ask subjects to "please indicate the dissimilarity between the stimulus pairs on a scale from 0 (identical) to 9 (maximally different)," then we enforce an upper bound on the responses (viz. 9) which conflicts with all minkowski metrics because it violates the Archimedean axiom on which they rest.

In Schönemann (1982) the hypothesis was advanced that a more plausible metric for such bounded dissimilarity data is given by the following metric for bounded response scales (MBR)

$$d_m = (a + b)/(1 + ab), \qquad 0 \leqslant a,\ b \leqslant 1, \tag{2.1}$$

where it will be assumed from now on that the data have been rescaled by division with the largest observed dissimilarity rating (plus a small constant to ensure all resulting relative distances fall into the half-open interval [0, 1)). This metric retains the simplicity of the city-block metric but respects the upper bound, 1. Thus the MBR is an adaptation of the well-known addition formula for hyperbolic tangents used in spacial relativity theory to take into account the empirical fact that no composite velocity can exceed the speed of light.

The psychological content of the MBR (2.1) is that the subject has to shorten the segments a, b that the stimulus interval c projects onto the subjective continua (see Fig. 1a) if he wants to produce city-block-like responses which fit into the response scale imposed by the experimenter, and that this distortion affects the larger segments more than the shorter segments. We thus have a second hypothesis which conflicts with the monotonicity hypothesis stated above:

MBR *hypothesis.* Dissimilarity judgments obtained with bounded response scales tend to be consistently biased towards underestimation of the larger distances. Hence no monotone transformation is required to restore the triangle inequality. However, if the subjects employ a city-block-like composition rule, then a known monotone transformation of the observed data may restore additivity.

No claim is made that subjects employ the above map literally at all times when confronted with bounded response scales. Rather, it is held that subjects will have to employ *some* contraction if they want to produce city-block-like numerical responses, and that in this case the hyperbolic tangent addition must be a better approximation to their judgments on purely logical grounds than any additive distance model.

Another simple concatenation which achieves such a contraction is $a + b - ab$. It is not difficult to verify that

$$a + b - ab \leqslant d_m \leqslant d_c \tag{2.2}$$

and that d_m never exceeds this lower bound by more than 0.05, so that this function might also be considered as a candidate to replace the city-block metric in the presence of upper bounds. However, there are some theoretical reasons which indicate that the MBR may be the most promising alternative among the simpler functions which respect the upper bound. In particular, the definition (2.1) implies that the MBR is intermediate between the two bounding metrics of the minkowski family, the city-block metric d_c and the sup-metric $d_s = \max\{a, b\}$,

$$d_s = \max\{a, b, \} \leqslant d_m = (a + b)/(1 + ab) \leqslant a + b = d_c. \tag{2.3}$$

The MBR (2.1) approximates d_c for distances close to zero, d_s for distances close to the upper bound 1, and the Euclidean distance d_e in the middle range. Thus d_m presents itself as an intuitively plausible explanation for why the otherwise psychologically unmotivated minkowski metrics usually give a reasonable fit for dissimilarity data. We therefore proceed with a more detailed analysis of the MBR and derive some empirical predictions which can be tested with dissimilarity data. For notational convenience, most of the discussion will be limited to the planar case, which is the most important case in practice. However, a closely related subadditive concatenation for norms can be obtained by applying the same logic to one dimension. This concatenation may be useful in psychophysics, where the upper bounds are often real.

3. Some Formal Properties of the MBR

As a first step, the three distance axioms will be checked.

Monotonicity. The MBR (2.1) is monotone increasing in both arguments separately.

To see this, suppose $a' \geqslant a$. Then $a'(1 - b^2) + aa'b + b \geqslant a(1 - b^2) + aa'b + b$, or equivalently, $a' + b + aa'b + ab^2 \geqslant a + b + aa'b + a'b^2$. Both sides can be factored to give $(a' + b)(1 + ab) \geqslant (a + b)(1 + a'b)$. Thus $a' \geqslant a = (a' + b)/(1 + a'b) \geqslant$

$(a + b)/(1 + ab)$, so that the MBR is monotone increasing in a. By symmetry this must also be true for b.

Triangle inequality. The MBR (2.1) satisfies the triangle inequality. This will be shown in 2 steps.

Step 1. Consider the particular point configuration in Fig. 2a, where the 3 vertices A, B, C project orthogonally in the order (A, B, C) onto the two coordinate axes. Let the projection of the sides $[A, B]$, $[B, C]$, $[A, C]$ onto the abscissa be a, p, r, respectively, and the corresponding projections onto the ordinate by b, q, s, as in Fig. 2a. Then $a + p = r$, $b + q = s \Rightarrow (a + b)/(1 + ab) + (p + q)/(1 + pq) \geqslant (a + b)/(1 + rs) + (p + q)/(1 + rs) = (r + s)/(1 + rs)$ so that the triangle inequality is met in this particular case. This argument remains valid for all fixed A, C and variable B as long as $a + p = r$ and $b + q = s$, i.e., geometrically, as long as B stays in the rectangle defined by the line segment $[A, C]$ as its diagonal with sides parallel to the coordinate axes. This includes the boundary cases where B coincides with one of the sides of this rectangles, as in Fig. 2b where $q = 0$ and $b = s$.

Step 2. Leaving A, C fixed, consider now a configuration where B falls outside the rectangle described in Step 1, so that the order of the projections of the 3 points onto the coordinate axes is permuted, e.g., the configuration in Fig. 2c. Denoting the projections of A, B, C onto the two coordinate axes as in Step 1, and those of A, B',

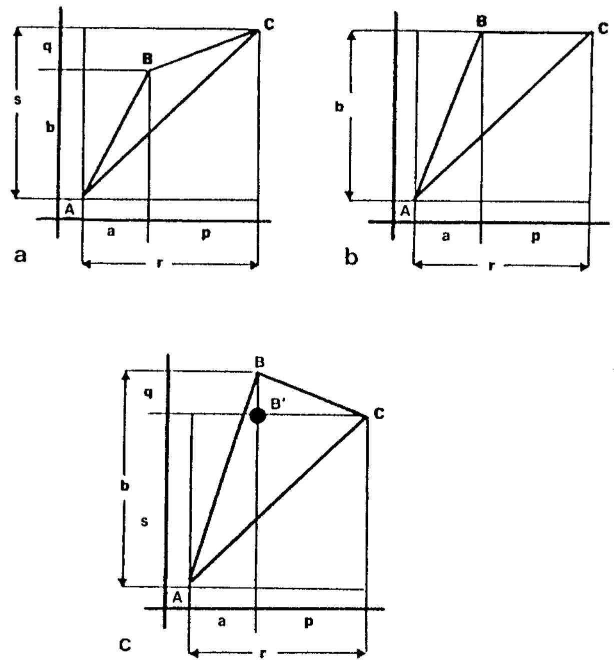

Fig. 2. Notation used in proof of the triangle inequality.

C, where B' is the orthogonal projection of B onto the side of the rectangle through C, by a', p', r', and b', q', s' one has $a = a'$, $p = p'$, $r = r'$, i.e., the projections onto the abscissa are the same for the triangles A, B, C and A, B', C, while for those onto the ordinate one has $b \geqslant b'$ $(=s)$, $q \geqslant q'$ $(=0)$, $s = s'$. Hence, by monotonicity,

$$(a+b)/(1+ab) + (p+q)/(1+pq) \geqslant (a+b')/(1+ab') + (p+q')/(1+pq')$$

and, by Step 1,

$$(a+b')/(1+ab') + (p+q')/(1+pq') \geqslant (r+s)/(1+rs)$$

so that the triangle inequality also holds for the configuration in Fig. 2c. Since the MBR is monotone in both arguments separately, the same reasoning can be repeated if B has to be moved both vertically and horizontally to obtain a B' on the boundary of the rectangle. Hence the triangle inequality is satisfied in the general case.

The present argument supercedes that given in Schönemann (1982), which is incomplete. The observation that the first step covers the worst possible case, and thus the gist of Step 2, is due to Professor H. Rubin, Department of Statistics, Purdue University.

Nonnegativity and symmetry of the MBR (2.1) are evident upon inspection. Whence the desired result:

The MBR (2.1) is a metric.

We now turn to the relation between the MBR and the minkowksi metrics. Inspection of the definition (2.1) shows that the MBR is close to d_c whenever the product ab is close to zero. Thus there is a strong though not perfect monotone relation between the city-block metric and the MBR over the range $0 \leqslant d_c \leqslant 1$. As one of the two arguments, say a, approaches the upper bound 1, the MBR approaches $(1+b)/(1+b) = 1$, and short of this bound it approaches $a = \max\{a, b\}$ if $a \geqslant b$, i.e., it behaves like the sup-metric for larger distances near 1. In view of the inequality (2.4), it must approach the intermediate minkowski metrics for intermediate distances, including the Euclidean distance. This behavior of the MBR is graphically illustrated in Fig. 3, where the "MBR-circles"

$$d_m = k, \qquad 0 \leqslant k \leqslant 1, \tag{3.1}$$

are sketched for various radii k. As can be seen, these "circles" resemble the unit disk for d_c for small k, that of d_s for large k, and they resemble Euclidean circles for raddi in the vicinity of 0.8. Thus, if the subjects were indeed attempting to produce city-block-like numerical assignments to the stimulus pairs in tasks with bounded response scales, which force them to employ the MBR instead, then the city-block metric should give a reasonable approximation because the percentage of smaller distances usually exceeds that of larger distances in a paired comparison task.

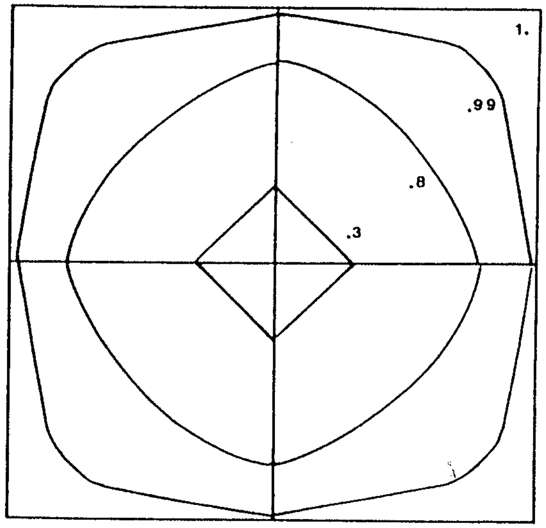

FIG. 3. MBR circles for various radii k.

However, since the larger distances are underestimated, originally straight lines will appear curved, as they often do (see, e.g., Krantz & Tversky, 1975, Fig. 4; Borg & Leutner, 1983, Fig. 8).

In terms of hyperbolic tangents the MBR can be written

$$\begin{aligned} d_m &= (a+b)/(1+ab) \\ &= [\tanh(u) + \tanh(v)]/[1 + \tanh(u)\tanh(v)], \\ &= \tanh(u+v), \end{aligned} \tag{3.2}$$

where $u = \tanh^{-1}(a)$, $v = \tanh^{-1}(b)$ are hyperbolic angles. Since they are additive, additivity of collinear line segments can be restored simply by transforming the relative distances with the inverse hyperbolic tangent transformation

$$\tanh^{-1}(d_m) = u + v. \tag{3.3}$$

If the MBR hypothesis is valid (which, of course, it need not be), then a subsequent analysis with the city-block metric should give a good fit and remove artefacts induced by the bounded response scales, such as unexpected curvature or logarithmic contractions.

A reviewer wondered about the relation between the MBR and the city-block metric: how different will d_c and d_m be in practice? To answer this question, we may proceed as follows:

The definition of d_c, d_m (Eqs (1.1) and (2.1)) imply that their ratio

$$r = d_c/d_m = 1 + ab, \qquad 0 \leqslant a, b < 1, \tag{3.4}$$

is maximized under the constraint that $d_c = \text{const.}$ when $a = = d_c/2$, since a square has largest area for rectangles of fixed circumference. Note that d_c varies between 0 and 2 as a, b, and hence d_m vary between 0 and 1. Hence

$$\max r = 1 + d_c^2/4, \qquad 0 \leqslant d_c = \text{const.} < 2, \tag{3.5}$$

which, in turn, implies that the largest difference between both metrics is

$$\max\{d_c - d_m\} = d_c - d_c/r = d_c^3/(4 + d_c^2). \tag{3.6}$$

Thus, the largest difference is 1 at the upper end of the d_c scale, and it is 0.2 when $d_c = 1$. At the 80% point of the d_c-scale ($d_c = 1.6$) the maximal difference between both metrics is 0.62.

4. Some Empirical Evidence Supporting the MBR Hypothesis

An empirical study was undertaken to check some of these predictions. Nine solid rectangles were presented on a computer in random pairings to 35 graduate students. The stimulus design was orthogonal in a metric width–weight coordinate system (heights: 1.1, 2.1, 3.1 cm; widths 2.7, 5.4, 8.1 cm). The rating scale varied from 0 ("identical") to 9 ("most dissimilar"). Since systematic difference between subjects were found on the basis of Hotelling T^2 tests, the data were analyzed for 3 groups (EQ group, 20 subjects; UP group, 9 subjects; DN group, 6 subjects) separately. The outcome of the MDS analysis will be reported in more detail elsewhere.

For the present article only one finding of this empirical study is relevant, because it has broader implications and held for all 3 groups equally: the contrasting predictions of violations of the triangle inequality by the MBR metric, the minkowski metrics and the monotonicity hypothesis.

Since the monotonicity hypothesis posits consistent violations of the triangle inequality, it predicts that larger distances will be consistently overestimated relative to smaller distances. This bias will be most pronounced for collinear segmental comparisons, which should be subadditive. In contrast, the MBR hypothesis predicts systematic underestimation of larger distances and hence superadditivity for collinear segments. Finally, all minkowski metrics imply segmental additivity for collinear comparisons parallel to the coordinate axes. The city-block metric further implies segmental additivity for diagonal comparisons.

One can therefore distinguish between these hypotheses on the basis of the proportion of violations of the triangle inequality in the observed data, and, if one is willing to assume that the underlying dimensions are known in advance, on the basis of collinear comparisons. In this case the sharpest test is provided by collinear comparisons which Attneave used to estimate his additive constant. Thus, let A, B, C be 3 points on a line with B between A and C, and let $d|A, B| = a$, $d|B, C| = b$ and

$d|A, C| = c$ be the distances between them (Fig. 1b). Then the mean of the simple function

$$f = a + b - c \tag{4.1}$$

will be negative if the triangle inequality is consistently violated, it will be positive if they are distances but subadditive, as the MBR predicts, and it will be zero for comparisons parallel to the coordinate axes if the distances are segmentally additive along these directions, as are all minkowski metrics:

(a) $E(f) = 0$ if violations of the triangle inequality are random and the data are minkowski distances,

(b) $E(f) < 0$ if the monotonicity hypothesis is true,

(c) $E(f) > 0$ if the MBR hypothesis is true.

The empirical distributions of f we obtained in our study are given in Fig. 4 for the 3

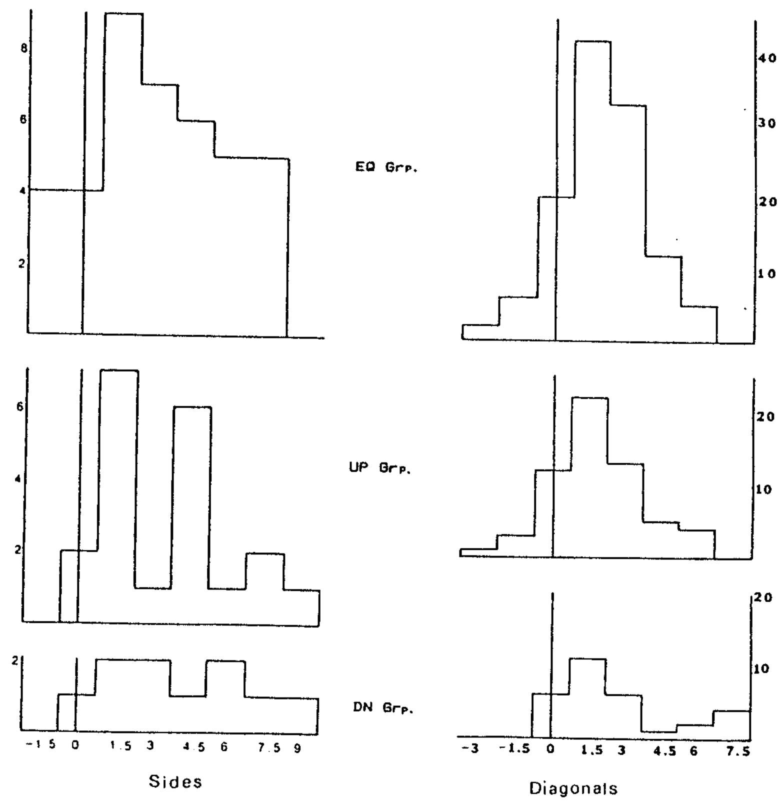

FIG. 4. Distributions of discriminant function $f = a + b - c$ for 3 groups of subjects, and for collinear comparisons of sides and diagonals separately. EQ group: 20 subjects; UP group: 9 subjects; DN group: 6 subjects.

groups of subjects and for the side comparisons and the diagonal comparisons separately. All 6 distributions have clearly positive means. This finding contradicts the monotonicity hypothesis, confirms that the data are distances, and it rules out all minkowski metrics. Finally, it is in agreement with the MBR hypothesis.

In conclusion, I wish to emphasize (a) that no claim is made that the MBR hypothesis will be true in general, (b) that it is limited to bounded scales (where the bounds could be subject-induced, however), (c) that it is predicated on the assumption that subjects attempt to use a city-block-like composition rule, and finally, (d) the condition $E(f) > 0$ is only necessary, not sufficient for the MBR hypothesis to be true. Corroborating evidence is provided, at least in theory, if the fit with city-block metric is markedly improved by the inverse tanh transformation. In practice, the definition of "marked improvement" may be difficult, however.

5. A Subadditive Concatenation for Norms with Natural Bounds

Viewed as a binary operation,

$$d_m = (a+b)/(1+ab) = a \oplus b \tag{5.1}$$

in the arguments a, b, one finds that $\oplus$ is symmetric and associative, has neutral element 0 and that an inverse operation exists in the half-open interval $[0, 1)$ given by

$$a \ominus b = (a-b)/(1-ab), \qquad 0 \leqslant a, b < 1. \tag{5.2}$$

If one applies the same logic used to motivate the MBR as a distance function to norms in one dimension, one obtains the subaddtive concatenation

$$n_{ij} \oplus n_{jk} = (n_{ij} + n_{jk})/(1 + n_{ij} n_{jk}), \qquad 0 \leqslant n_{ij} \leqslant 1 \tag{5.3}$$

for the norms $n_{ij} = |x_j - x_i|$ of line segments separating three ordered points $i \leqslant j \leqslant k$ on a 1-dimensional continuum x.

This concatenation rule predicts underestimation for doubling,

$$x \oplus x = d(x), \tag{5.4}$$

$$d(x) = 2x/(1+x^2), \qquad 0 \leqslant x \leqslant 1, \tag{5.5}$$

overestimation for halving,

$$h(x) \oplus h(x) = x, \tag{5.6}$$

$$h(x) = [1 - (1-x^2)^{1/2}]/x, \qquad 0 \leqslant x \leqslant 1, \tag{5.7}$$

and overestimation for middling,

$$m(x, y) \oplus m(x, y) = x \oplus y, \tag{5.8}$$

$$m(x, y) = [1 + xy - (1-x^2)^{1/2}(1-y^2)^{1/2}]/[x+y], \qquad 0 \leqslant x \leqslant y \leqslant 1. \tag{5.9}$$

The halving function is convex and the doubling function concave over the interval $[0, 1)$. Hence, if one were to approximate these functions with power laws, then the exponent for halving would be larger than 1 and the exponent for doubling smaller than 1 (Fig. 5).

In Schönemann (1982) it was suggested that this concatenation may be appropriate for psychophysical data if the responses are subject to natural upper bounds. An example is velocity perception, where subjects are incapable of discriminating between velocities which exceed a certain upper limit. As Caelli, Hoffman, & Lindman (1978) found, this upper limit varies from subject to subject. This examples is, of course, very close in spirit to the precedent from physics. Caelli *et al.* (1978) pursued the analogy still further by computing so-called "Lorenz transformations" to account for their data. Formally, these transformations are motions in minkowski spaces, that is, in the simplest, planar case, 1-parameter transformations which preserve hyperbolae as unit disks. Caelli *et al.* (1978) calculated these motions as a funciton of the highest speed a subject was still able to distinguish, and then used them to predict length contractions obtained in a separate experiment.

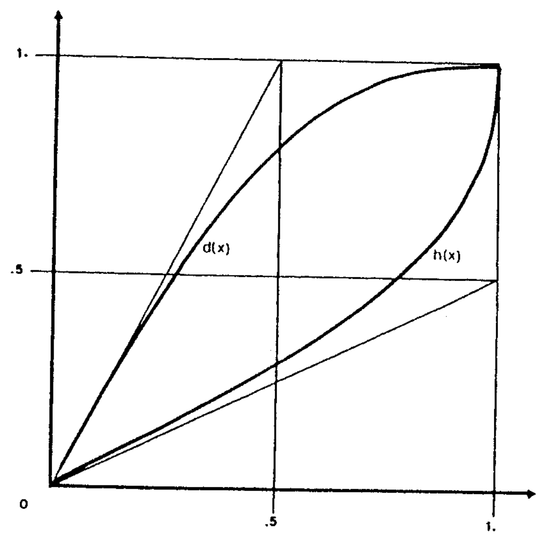

FIG. 5. Doubling function $d(x)$ (Eq. (5.5)) and halving function $h(x)$ (Eq. (5.7)) implied by 1-dimensional concatenation (5.3). The tangents are the veridical doubling and halving functions: $d(x)$ underestimates and $h(x)$ overestimates.

In loc. cit. it was further argued that even when the connection with physics is less immediate than in the Caelli *et al.* (1978) study, the analogous concatenation rule for norms (5.1) may be useful whenever the continuum involved can be expected to be bounded above, and the bounds are natural, rather than experimenter imposed. The psychological motivation for this expectation is essentially the same as was used to motivate the MBR. Similarly, preprocessing of the data with the hyperbolic tangent transformation may then often remove seemingly bizarre distortions of psychophysical functions if they are caused by natural bounds.

The main point is simply that bounds, whether real or experimenter imposed, rule out additive structures because they conflict with the Archimedean axiom.

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